1. (12 marks) Solve the initial value problem

$$\sin(x)\frac{dy}{dx} + 2y\cos(x) = 4\cos(x), \quad y(\pi/2) = 0.$$

   On what interval is this solution valid?

2. (10 marks) Solve implicitly

$$(x^3 + y^3)dx + (2xy^2 - \frac{1}{4}x^4y^{-1} + 2y^3)dy = 0$$

3. (12 marks) Solve the initial value problem

$$y'' - 4y' + 5y = 2\sin x, \qquad y(0) = 0, \quad y'(0) = 0.$$

4. (10 marks) Find the general solution of

$$x^2y'' + xy' - y = 2x^2\cos x, \qquad x > 0.$$

5. (12 marks) Find the general solution of

$$\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = 1 + 9e^{2x}.$$

6. (10 marks) Using Laplace transforms, and making use of the table below as needed, solve

$$y'' + 2y = \delta(t - \pi)$$

   with initial conditions $y(0) = 1$, $y'(0) = 2$.

function $f(t)$	Laplace transform $F(s)$
1	$1/s \ (s > 0)$
t^n	$n!/s^{n+1} \ (s > 0)$
e^{at}	$1/(s-a) \ (s > a)$
$\sin at$	$a/(s^2 + a^2) \ (s > 0)$
$\cos at$	$s/(s^2 + a^2) \ (s > 0)$
$e^{-at}f(t)$	$F(s + a)$
$\mathcal{U}(t - a)$ or $\mathcal{U}_a(t)$ $(a \geqslant 0)$	$e^{-as}/s \ (s > 0)$
$\delta(t - a) \ (a > 0)$	e^{-as}
$\mathcal{U}(t-a)f(t-a)$ or $\mathcal{U}_a(t)f(t-a)$	$e^{-as}F(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{(n-1)}(0)$
$f * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$

7. (12 marks) Consider the following matrix A and its row reduced echelon form R

$$A = \begin{pmatrix} 2 & 0 & -1 & 1 & 4 \\ 2 & 0 & 3 & 5 & 12 \\ 1 & 0 & 1 & 2 & 5 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   (a) Write down a basis for the row space of A, and a basis for the column space of A.

   (b) Find a basis for the null space (also known as the kernel) of A.

   (c) Find an orthogonal basis for the column space of A.

8. (10 marks) Find the eigenvalues and corresponding eigenvectors of the symmetric matrix

$$A = \begin{pmatrix} 2 & 4 \\ 4 & -4 \end{pmatrix}$$

Find an orthogonal matrix P with $A = PDP^T$, and D diagonal.

Compute the matrices in the standard basis of orthogonal projection onto each eigenspace.

9. (12 marks) The matrix $A = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$ has an eigenvalue $\lambda = 5 + 2i$ and corresponding eigenvector $\begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix}$

(a) Write down a basis, in vector form, of real solutions of the system

$$\begin{aligned} x_1' &= 6x_1 - x_2 \\ x_2' &= 5x_1 + 4x_2 \end{aligned}$$

(b) Find functions $x_1(t)$ and $x_2(t)$ which satisfy the system above as well as the initial conditions $x_1(0) = 1$, $x_2(0) = 0$.

(c) How do solutions of the system behave as $t \to -\infty$?